# A self-similar dodecagonal quasiperiodic tiling of the plane in terms of squares, regular hexagons and thin rhombi



1988 J. Phys. A: Math. Gen. 21 2167

(http://iopscience.iop.org/0305-4470/21/9/029)

J. Phys. A: Math. Gen. **21** (1988) 2167-2175. Printed in the UK

# A self-similar dodecagonal quasiperiodic tiling of the plane in terms of squares, regular hexagons and thin rhombi

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Received 4 August 1987, in final form 21 October 1987

**Abstract.** We present a dodecagonal quasiperiodic tiling of the plane in terms of three kinds of tiles: a square, a regular hexagon and a thin rhombus whose acute inner angles are 30°. The dodecagonal tiling is self-similar with respect to a dilatation by $2+\sqrt{3}$. It is associated with a dodecagonal quasiperiodic lattice which is obtained by a projection from a hyperhoneycomb lattice in four dimensions. The hyperhoneycomb lattice is a non-Bravais lattice composed of four Bravais sublattices and we must assign 'windows' with different orientations to different sublattices. We discuss in detail the relationships of the present method of obtaining a dodecagonal tiling to other methods.

## 1. Introduction

Since the discovery of a dodecagonal quasicrystal (Ishimasa *et al* 1985), several dodecagonal quasiperiodic lattices (quasilattices) in 2D (two dimensions) were proposed as models of the quasicrystal (Stampfli 1986, Niizeki and Mitani 1987, Ishihara 1987, Yang and Wei 1987). Some of the dodecagonal quasilattices (DQL) are obtained with the projection method (for the projection method, see Krammer and Neri (1984), Kalugin *et al* (1985), Duneau and Katz (1985), Janssen (1986) and Gähler and Rhyner (1986)) from a 4D hyperhexagonal lattice (Niizeki and Mitani 1987, hereafter referred to as I). The DQL are self-similar with respect to a composite transformation between a dilatation by $\tau_p=(\sqrt{3}+1)/\sqrt{2}$, the 'platinum ratio' introduced in I, and a rotation by 15°. They give rise to tilings of the plane. The basic tiles of one of the tilings are a regular triangle, a square and a thin rhombus whose acute inner angles are 30°.

On the other hand, Ishihara (1987) and, independently, Yang and Wei (1987) reported that DQL can be obtained with the projection method from a 6D simple hypercubic lattice. The basic tiles of the tiling associated with the resulting dodecagonal quasilattice are the square, the thin rhombus and a thick rhombus whose acute inner angles are 60°. The thick rhombus is identical to a union of the two regular triangles.

In this paper we will present a dodecagonal quasiperiodic tiling of the plane in terms of the thin rhombi, the squares and regular hexagons, each of which is identical to a union of the three thick rhombi. The tiling is self-similar with respect to a simple dilatation by $2+\sqrt{3}$. The tiling is associated with a dodecagonal quasiperiodic lattice obtained by a projection from a 4D hyperhoneycomb lattice, which is the direct product of two 2D honeycomb lattices. The 4D lattice is not a Bravais lattice and no quasilattices have been obtained previously from a non-Bravais lattice.

Since the DQL associated with the present dodecagonal quasiperiodic tiling is constructed in a similar way as that by which DQL are constructed in I from a 4D hyperhexagonal lattice, we summarise in § 2 the construction in I. In § 3, we investigate in detail properties of the 4D hyperhoneycomb lattice, especially its relationship with the 4D hyperhexagonal lattice. In § 4, we present a construction of a DQL by a projection from the 4D hyperhoneycomb lattice. We investigate in § 5 a dodecagonal quasiperiodic tiling associated with the DQL constructed in the preceding section. We investigate in § 6 self-similarity of the dodecagonal quasiperiodic tiling. In the final section, § 7, we discuss related subjects, in particular, the relationships of the present method of obtaining a dodecagonal tiling with other methods.

## 2. Summary of the construction of DQL in I

The starting lattice is a 4D hyperhexagonal lattice, $L_{\rm HH}$, which is a direct product of two identical triangular lattices in 2D. The plane (a 2D Euclidean space) is identified with the complex plane and a 2D triangular lattice, $L_{\rm T}$, with the set of all the triangular integers (Eisenstein integers), $\boldsymbol{Z}[\omega]=\{n_1+n_2\omega|n_1, n_2\in\boldsymbol{Z}\}$, where $\omega=\exp(2\pi i/3)$ $(=\frac{1}{2}(-1+\sqrt{3}i))$. Then, $L_{\rm HH}=L_{\rm T}\times L_{\rm T}$ is identified with a complex 2D lattice as $L_{\rm HH}\simeq\{(\nu_1,\nu_2)|\nu_1,\nu_2\in\boldsymbol{Z}[\omega]\}$, which is included in a complex 2D space, $\boldsymbol{C}^2(=\boldsymbol{C}\times\boldsymbol{C})=\{(z_1,z_2)|z_1,z_2\in\boldsymbol{C}\}$.

A $2\times2$ complex matrix defined by

$$R=\begin{pmatrix}0 & \zeta^2\\ 1 & 0\end{pmatrix} \tag{1}$$

with $\zeta=\exp(\frac{1}{6}\pi i)$ $(=\frac{1}{2}(\sqrt{3}+i))$ is a unimodular unitary matrix of $\boldsymbol{Z}[\omega]$, an integral domain, because $\zeta^2=1+\omega$ is a triangular integer and $\det R=-\zeta^2$, which is a unit of the integral domain. Therefore, linear transformation $r$ represented by $R$ is a point symmetry of $L_{\rm HH}$; $r$ is an orthogonal transformation which transforms $L_{\rm HH}$ into itself. Since $R^6=-I$ and $R^{12}=I$ with $I$ being the unit $2\times2$ matrix, the order of $r$ is 12.

The eigenvalues of $R$ are $\zeta$ and $-\zeta$ and $\boldsymbol{C}^2$ decomposes into the corresponding two invariant complex-1D subspaces, $\boldsymbol{C}_+$ and $\boldsymbol{C}_-$, which are called the external space and the internal one, respectively. Note that both the subspaces can be identified with 2D Euclidean spaces. The projections of $(z_1,z_2)\in\boldsymbol{C}^2$ onto $\boldsymbol{C}_+$ and $\boldsymbol{C}_-$ are given, apart from a numerical factor, by $z_\pm=z_1\pm\zeta z_2$ because $(1,\pm\zeta)$ are two row eigenvectors of $R$ corresponding to eigenvalues $\pm\zeta$. The action of $r$ onto $\boldsymbol{C}^2$ decomposes into two linear transformations, $z_\pm\to\pm\zeta z_\pm$, which are rotations of $\boldsymbol{C}_\pm$.

A dodecagonal quasilattice (DQL) is a set of points in $\boldsymbol{C}_+$ and given by $Q_{\rm D}=\{\nu_1+\zeta\nu_2|(\nu_1,\nu_2)\in L_{\rm HH}$ and $\nu_1-\zeta\nu_2+\phi\in W\}$, where $\phi$ is a complex number and $W$, the window, is a finite domain in $\boldsymbol{C}_-$.

## 3. The hyperhoneycomb lattice in 4D

A triangular lattice can be decomposed into three identical sublattices which are also triangular lattices but with lattice constants being $\sqrt{3}$ times that of the original one. To be more specific, the sublattices are labelled by numbers in the Galois field $\boldsymbol{Z}_3\equiv\{0,1,2\}=\boldsymbol{Z}/3\boldsymbol{Z}$;

$$L_{\rm T}=\bigcup_{p\in\boldsymbol{Z}_3}L_{\rm T}(p) \tag{2}$$

$$L_{\rm T}(p)=\{n_1+n_2\omega|n_1,n_2\in\boldsymbol{Z}\text{ and }n_1+n_2\equiv p\bmod 3\}.$$

A multiplication of $\zeta^2$ onto $\boldsymbol{C}$ gives rise to a rotation of $\boldsymbol{C}$ by 60°, so that $L_\mathrm{T}$ is invariant against the multiplication; $\zeta^2 L_\mathrm{T} = L_\mathrm{T}$†. It transforms the three sublattices of $L_\mathrm{T}$ into themselves as $\zeta^2 L_\mathrm{T}(0) = L_\mathrm{T}(0)$, $\zeta^2 L_\mathrm{T}(1) = L_\mathrm{T}(2)$ and $\zeta^2 L_\mathrm{T}(2) = L_\mathrm{T}(1)$. Therefore, the union of two sublattices, $L_\mathrm{T}(1)$ and $L_\mathrm{T}(2)$, is invariant against the multiplication. The union is nothing but the honeycomb lattice, $L_\mathrm{HC}$. That is, $L_\mathrm{HC} = L_\mathrm{T}(1) \cup L_\mathrm{T}(2)$ and $\zeta^2 L_\mathrm{HC} = L_\mathrm{HC}$. An important difference of $L_\mathrm{HC}$ from $L_\mathrm{T}$ is that it is not a Bravais lattice. We remark here for later convenience the relationships, $\zeta^2 L_\mathrm{T}(p) = L_\mathrm{T}(-p)$ with $p = 1$ or 2, because $-1 \equiv 2$ and $-2 \equiv 1$ modulo 3.

The triangular lattice $L_\mathrm{T}(0)$ naturally gives rise to a partition of the plane into regular triangles. Each triangle is nothing but the Voronoi polygon (Wigner–Seitz cell) of a lattice point of $L_\mathrm{HC} = L_\mathrm{T}(1) \cup L_\mathrm{T}(2)$. We shall call such a division (of the plane) a Voronoi partition associated with $L_\mathrm{HC}$. Mathematically, a Voronoi partition is a set of the points on the boundaries between different Voronoi cells.

It is well known that $L_\mathrm{T}$ is obtained from a simple cubic lattice, $L_\mathrm{SC}$, by projecting it along the [111] direction; the 2D lattices of (111) lattice planes of $L_\mathrm{SC}$ are projected to the three triangular sublattices of $L_\mathrm{T}$ because they stack along [111] like $\ldots ABCABC \ldots$.

Since $L_\mathrm{T}$ consists of three triangular sublattices, $L_\mathrm{HH}$ ($= L_\mathrm{T} \times L_\mathrm{T}$) are divided into nine sublattices, which are 4D hyperhexagonal lattices but with a lattice constant being $\sqrt{3}$ times that of $L_\mathrm{HH}$. The nine sublattices are labelled by pairs of elements in $\boldsymbol{Z}_3$. Alternatively, they are labelled by numbers in the Galois field $\boldsymbol{Z}_3[i] \equiv \{p + iq \mid p, q \in \boldsymbol{Z}_3\}$ ($= \boldsymbol{Z}[i]/3\boldsymbol{Z}[i]$ with $\boldsymbol{Z}[i]$ being the integral domain of Gaussian integers) as $L_\mathrm{HH}(\lambda)$, $\lambda \in \boldsymbol{Z}_3[i]$. The nine sublattices of $L_\mathrm{HH}$ are transformed by $r$ to themselves. We can easily confirm from equations (1) and (2) and the relationships, $\zeta^2 L_\mathrm{T}(q) = L_\mathrm{T}(-q)$, that $L_\mathrm{HH}(\lambda)$ with $\lambda = p + iq$ $\boldsymbol{Z}_3[i]$ is transformed to $L_\mathrm{HH}(\lambda')$ with $\lambda' = -q + ip = i\lambda$ ($\in \boldsymbol{Z}_3[i]$).

Note that $L_\mathrm{HH}$ is isomorphous to the 4D lattice obtained from a 6D simple hypercubic lattice, $L_\mathrm{SHC}$($= L_\mathrm{SC} \times L_\mathrm{SC}$), by projecting it onto a 4D subspace, $\mathrm{E}_4$, which is orthogonal to the [111000] and [000111] directions; the nine sublattices of $L_\mathrm{HH}$ are the projections of the 4D lattices of parallel 4D lattice planes of $L_\mathrm{SHC}$ to $\mathrm{E}_4$.

A 4D hyperhoneycomb lattice, $L_\mathrm{HHC}$, is defined as the direct product of two 2D honeycomb lattices, $L_\mathrm{HHC} = L_\mathrm{HC} \times L_\mathrm{HC}$. It is decomposed into four identical sublattices, which are labelled by numbers in $\Lambda = \{1 + i, 2 + i, 2 + 2i, 1 + 2i\} \subset \boldsymbol{Z}_3[i]$. $r$ is a point symmetry, also, of $L_\mathrm{HHC}$ and the four sublattices are transformed into themselves as $rL_\mathrm{HH}(\lambda) = L_\mathrm{HH}(i\lambda)$ (note that $\Lambda = i\Lambda$).

The Vonoi polytope of a lattice point, $(\nu_1, \nu_2)$, of $L_\mathrm{HHC} = L_\mathrm{HC} \times L_\mathrm{HC}$ is a hypertrigonal prism, $V = T_1 \times T_2 = \{(z_1, z_2) \mid z_1 \in T_1, z_2 \in T_2\}$, where $T_1$ (or $T_2$) is a Voronoi triangle of a lattice point, $\nu_1$ (or $\nu_2$), of $L_\mathrm{HC}$. Therefore, the Voronoi partition (of the 4D Euclidean space) associated with $L_\mathrm{HHC}$ is given by $(\Pi_\mathrm{HC} \times \mathrm{E}_2) \cup (E_2 \times \Pi_\mathrm{HC})$, where $E_2$ is a 2D Euclidean space and $\Pi_\mathrm{HC}$ the Voronoi partition associated with the $L_\mathrm{HC}$.

## 4. A DQL by a projection of a 4D hyperhoneycomb lattice

Since $r$ is a point symmetry, also, of $L_\mathrm{HHC}$, we may construct a DQL by a projection of $L_\mathrm{HHC}$ onto $\boldsymbol{C}_+$, the external space. It is important at this point how to choose the

† Let $c$ be a number and let $X = \{x, x_2, \ldots\}$ be a set of numbers. Then $cX$ is a set of numbers defined by $cX = \{cx_1, cx_2, \ldots\}$.

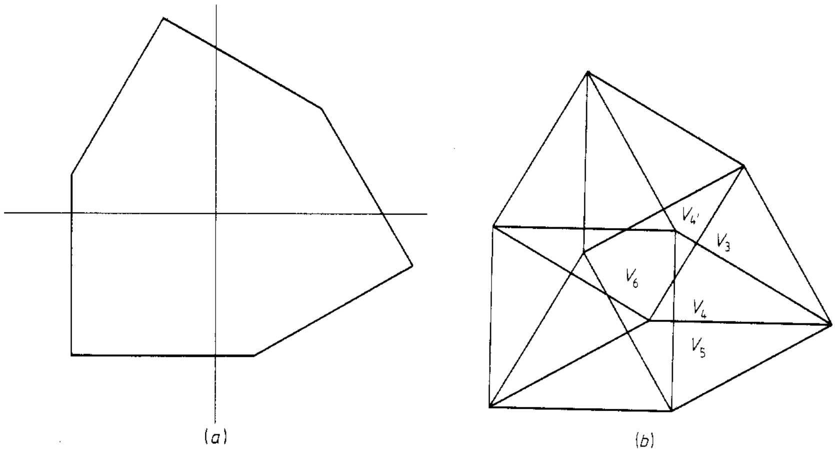


**Figure 1.** ($a$) A window of a trigonal hexagon, which is the projection of a Voronoi polytope of $L_{\mathrm{HHC}}$ onto $C_{-}$. ($b$) A division of the window in ($a$) into subwindows, which are associated with the five types of vertices.

window in $C_{-}$, the internal space. A most natural choice is the projection of a Voronoi polytope of $L_{\mathrm{HHC}}$. There is a complication here because $L_{\mathrm{HHC}}$ is not a Bravais lattice; the projections of the Voronoi polytopes of lattice points belonging to different sublattices have different orientations in the internal space, $C_{-}$, although they have the same shape and size. Therefore, windows with different orientations are assigned to different sublattices of $L_{\mathrm{HHC}}$. They are labelled by numbers in $\Lambda$ as $W(\lambda)$, $\lambda \in \Lambda$. Since $rL_{\mathrm{HHC}}(\lambda) = L_{\mathrm{HHC}}(i\lambda)$ for $\forall \lambda \in \Lambda$ and $r$ reduces in $C_{-}$ to a multiplication of $-\zeta$, we obtain $-\zeta W(\lambda) = W(i\lambda)$ for $\forall \lambda \in \Lambda$. These relationships fix relative orientations among the windows. The shape and the size of a representative window, $W(1+i)$, is obtained by projecting $T \times T$ onto $C_{-}$, where $T$ is a regular triangle whose vertices are at $\zeta^2$, $-1$ and $\zeta^{-2}$ on $C_{-}$. It is given by $W(1+i) = \{z_1 - \zeta z_2 | z_1 \in T, z_2 \in T\}$, which is an equisided trigonal hexagon, as shown in figure 1($a$). Its smaller three inner angles are 90° and the larger three 150°. The length of the six sides is equal to $\surd 3$. The positions of the three vertices with inner angles being equal to 150° are at $1+i$, $(1+i)\omega$ and $(1+i)\omega^2$ on $C_{-}$.

We can now write down a DQL as

$$Q_{\mathrm{D}}(\phi) = \bigcup_{\lambda \in \Lambda} \{\nu_1 + \zeta\nu_2 | (\nu_1, \nu_2) \in L_{\mathrm{HH}}(\lambda) \text{ and } \nu_1 - \zeta\nu_2 + \phi \in W(\lambda)\} \tag{3}$$

where $\phi$ is a complex number representing the 'phase vector'.

## 5. A dodecagonal quasiperiodic tiling associated with a DQL

If a pair of sites in a 2D quasilattice are projections of a pair of nearest neighbours in the starting higher-dimensional lattice, we call it an arithmetic neighbour pair (Katz and Duneau 1986). We can assign a bond to each part of such a pair. Then the bonds form a 2D quasiperiodic network or, equivalently, give rise to a quasiperiodic tiling of the plane. Each site of $L_{\mathrm{HC}}$ has three nearest neighbours and that of $L_{\mathrm{HHC}}$ ($= L_{\mathrm{HC}} \times L_{\mathrm{HC}}$) has six ($= 3+3$). A site in the DQL presented in the previous section has six arithmetic neighbours at most and a bond can assume, by equation (3), one of twelve orientations,

$1, \zeta, \zeta^2, \ldots, \zeta^{11}$. The DQL has only five types of vertices, which will be identified easily in the tiling to be presented shortly. They are associated with the subwindows to which the original windows are divided as in fiugre 1(*b*); subwindows which are congruent with each other correspond to vertices which differ only in their orientations (for the principle of dividing a window into subwindows, see Katz and Duneau (1986)). The coordination numbers of the five vertices range from three to six and we denote them as $V_3$, $V_4$, $V_{4'}$, $V_5$ and $V_6$, respectively; both of $V_4$ and $V_{4'}$ have coordination number 4. A vertex has the same point symmetry as that of the associated subwindow; $V_4$ and $V_{4'}$ have a mirror symmetry and $V_6$ a trigonal symmetry. The frequencies of appearances of different types of vertices are proportional to the areas of the associated subwindows: the probabilities of appearances of $V_3$, $V_4$, $V_{4'}$, $V_5$ and $V_6$ are calculated to be $3c\tau_p^{-1}$, $3\sqrt{3}c\tau_p^{-3}$, $\sqrt{3}\ c\tau_p^{-3}$, $3c\tau_p^{-5}$ and $c\tau_p^{-4}$, respectively, with $c = 1/2\sqrt{2}$ and $\tau_p \equiv (\sqrt{3}+1)/\sqrt{2}(=2\cos(\pi/12))$.

A portion of the dodecagonal quasiperiodic tiling discussed above is shown in figure 2. There are only three kinds of tiles; a square ($T_S$), a regular hexagon ($T_H$) and a thin rhombus ($T_R$) whose acute inner angles are 30°. The appearance of hexagonal tiles is due to the fact that $L_{HHC}$ includes 2D honeycomb lattices as its lattice planes. Hexagons take two orientations corresponding to the two terms in $\nu_1 + \zeta\nu_2$ or, equivalently, to the two series of 2D lattice planes with honeycomb structure. On the other hand, squares and rhombi are related to projections of other series of 2D lattice planes with a square lattice structure.

A notable feature of the tiling is that two tiles of the same kind never neighbour side by side. This appears to be the sole principle of the tiling but we cannot prove it yet. The relative frequencies of appearances of the three kinds of tiles, $T_S$, $T_H$ and $T_R$, to that of the sites (vertices) are calculated from the vertex statistics to be $c\sqrt{3}\tau_p^{-1}$, $c\tau_p^{-1}$ and $c\sqrt{3}\tau_p^{-1}$, respectively, with $c = 1/2\sqrt{2}$.

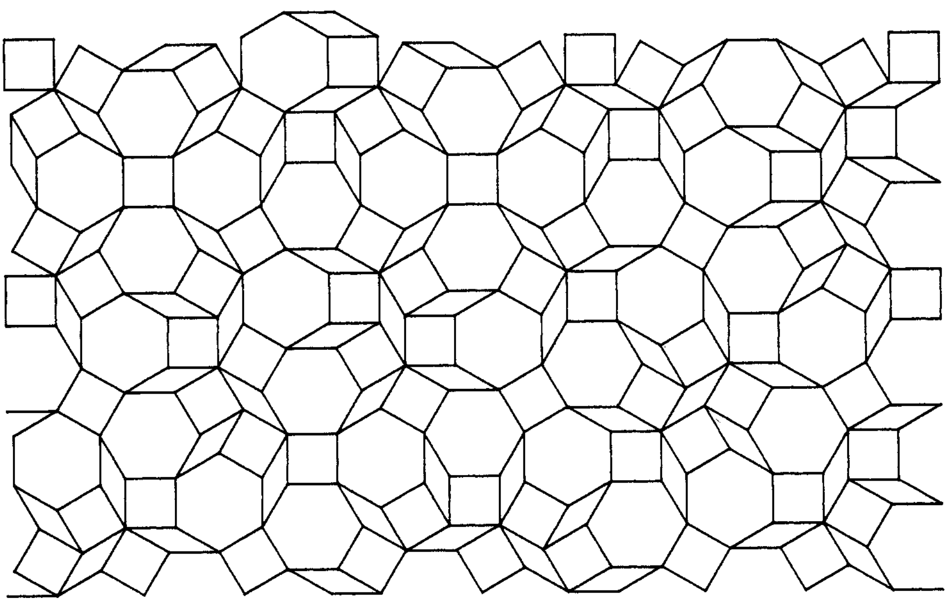

**Figure 2.** A docecagonal quasiperiodic tiling obtained by a projection from a 4D hyper-honeycomb lattice. There are two types of vertices, $V_4$ and $V_{4'}$, whose coordination numbers are four. $V_4$ is a vertex at which two thin rhombi and two squares meet.

The basic tiles of the present tiling are only polygons with even number of vertices in contrast to the case in I. Therefore, the present DQL can be divided into two interconnected sublattices. In fact, it is divided into four sublattices which are labelled by numbers in $\Lambda$ on the basis of their correspondence to the sublattices of $L_{\mathrm{HH}}$; two sublattices labelled by two consecutive numbers in series $1+i \to 2+i \to 2+2i \to 1+2i \to 1+i$ are bonded with each other but not in other cases. The four vertices of a square or a rhombus belong to the four different sublattices from one another. The six vertices of a hexagon belong to two sublattices labelled by two consecutive numbers in the above series.

## 6. Self-similarity of the dodecagonal quasiperiodic tiling

The self-similarity of a quasilattice and that of the associated quasiperiodic tiling are connected with a linear transformation which is an automorphism (one-to-one mapping onto itself) of the starting lattice (Katz and Duneau 1986, Gähler 1986). The linear transformation used in I is the one represented by the unimodular matrix, $M = I + R$, whose eigenvalues are $1+\zeta$ and $1-\zeta$. It is an automorphism of $L_{\mathrm{HH}}$ and transforms the nine sublattices into themselves as $L_{\mathrm{HH}}(\lambda) \to L_{\mathrm{HH}}((1+i)\lambda)$ ($\lambda \in \boldsymbol{Z}_3[i]$). Unfortunately, it is not an automorphism of $L_{\mathrm{HHC}}$ because it does not transform the four sublattices into themselves. Thus, the DQL presented in this paper is not self-similar with respect to the composite transformation between a dilatation by $\tau_p = |1+\zeta|$ and a rotation by $\arg(1+\zeta)$, in contrast to the DQL in I.

On the other hand, a linear transformation represented by $M' = 2I + R + R^{-1}$ $(= R^{-1}M^2)$ is an automorphism of $L_{\mathrm{HHC}}$ because its sublattices are transformed to themselves as $L_{\mathrm{HH}}(\lambda) \to L_{\mathrm{HH}}(2\lambda)$ (note that $2\Lambda = \Lambda$). It decomposes into a dilatation of $\boldsymbol{C}_+$ by $\tau_p^2 = 2+\sqrt{3}$ $(=2+2\cos(\frac{1}{6}\pi))$ and a contraction of $\boldsymbol{C}_-$ by $\tau_p^{-2} = 2-\sqrt{3}$. Moreover, it can be shown easily that $W(2\lambda) = -W(\lambda)$ and $\tau_p^{-2}W(2\lambda) = -\tau_p^{-2}W(\lambda)$, which is identical to the central subwindow of $W(\lambda)$ as seen in figure 1(*b*). From these, we can conclude as in I that the present DQL and the associated quasiperiodic tiling are self-similar with respect to a dilatation by $\tau_p^2$.

We show in figure 3 the once inflated tiling which is superposed onto the original one. The inflated quasilattice is obtained by leaving only the vertices of type $V_6$ in the original lattice, for the reason mentioned in the last paragraph. If the tiling represented by the bold lines in figure 3 is regarded to be the original tiling, the thin one is its deflated version. The rules of decorating $T_{\mathrm{S}}$ and $T_{\mathrm{R}}$ on the deflation are obvious from figure 3. On the other hand, the one for $T_{\mathrm{H}}$ is complicated. We can find it by scrutinising figure 3. We will not, however, mention it here but leave it to the reader for his exercise.

## 7. Discussions

The Fourier transform of a quasilattice constructed by the projection method is easily calculated (Zia and Dallas 1985, Katz and Duneau 1986) and we can obtain the diffraction pattern. The positions of the Bragg spots are common for all the DQL except scaling factors which are related to the 'lattice constants'. However, the relative intensities among different spots of a particular DQL depend on the LI class to which it belongs. The diffraction pattern of a DQL presented in I has a 'self-similarity' characterised by complex number $\tau = 1+\zeta (= \tau_p \exp(\frac{1}{12}\pi \mathrm{i}))$; a series of spots with

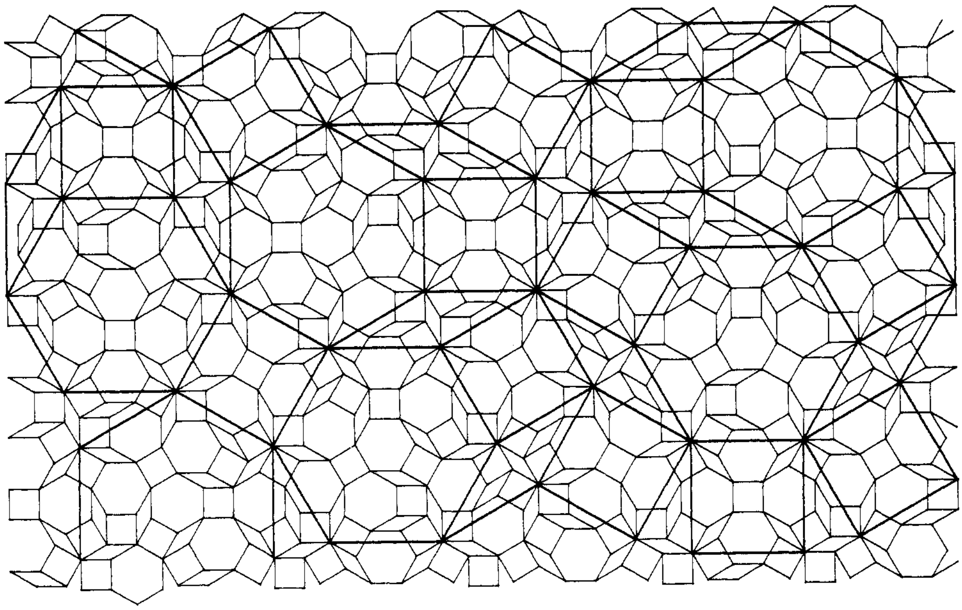

**Figure 3.** An original dodecagonal quasiperiodic tiling (thin lines) and its inflated version.

increasing intensities are arranged spirally at $\tau^n$ ($n = 1, 2, 3, \ldots$) times the position of the starting spot, where the reciprocal space is identified with a complex plane. On the other hand, we can expect for the diffraction pattern of the present DQL only that the spots are arranged radially because its self-similarity is not characterised by $\tau$ but by $\tau_p^2 (= |\tau|^2)$. Apart from this point, the diffraction pattern of the present DQL is similar to the one in I and we will not present the figure in this paper.

The LI class of the DQL in this paper has been obtained by including only four sublattices of $L_{\mathrm{HH}}$ among the nine. We may obtain different LI classes of DQL if other sublattices are included and different windows are assumed for different sublattices; the windows must be consistent with a macroscopic dodecagonal symmetry. Of the LI classes, there is one which is closely related to the one presented in the present paper; a DQL belonging to the former LI class is obtained from the latter by adding lattice points to all the centres of the hexagonal tiles. The additional lattice points come from four sublattices, $L_{\mathrm{HH}}(i^k)$, $k = 0$–$3$, of $L_{\mathrm{HH}}$ and the windows to be assigned to the four sublattices are regular triangles with sides of length being equal to $\sqrt{3}$. The three vertices of a representative window, say, $W(1)$, are at 1, $\omega$ and $\omega^2$ on $\boldsymbol{C}_-$. Other three windows are related to $W(1)$ as $W(i^k) = (-\zeta)^k W(1)$, $k = 1$-$3$. This LI class is, alternatively, obtained by a projection from $L_{\mathrm{SHC}}$, the 6D simple hypercubic lattice; the close connection between the two ways, i.e. the projection from $L_{\mathrm{HH}}$ and the one from $L_{\mathrm{SHC}}$, will be understandable by the relationship between the two lattices, as mentioned in § 3. The LI class is, however, different from the one obtained by Ishihara (1987) with the projection method from $L_{\mathrm{SHC}}$; the maximum coordination number of the former LI class is six but the one for the latter is twelve. The LI class obtained by Ishihara can be derived by the present method if different windows from the above choices are used.

Quasiperiodic tilings of the plane can be obtained, alternatively, by using the grid method (de Bruijn 1981, Gähler and Rhyner 1986, Stampfli 1986). In this method, a quasiperiodic tiling is obtained as a dual tiling to a quasiperiodic grid, which is usually a superposition of two or more periodic grids. It was proved generally by Gähler and Rhyner (1986) that the projection method and the grid method are equivalent in the cases where the basic tiles are rhombi. We can generalise this result to the cases where some of the basic tiles are not rhombi as the present case; a quasiperiodic tiling obtained by a projection from a higher-dimensional lattice is obtained also by the grid method if the window(s) used in the projection method is the projection(s) of the relevant Voronoi polytope(s) of the starting lattice onto the internal space. Then, the grid to be used in the grid method is the cross section of the Voronoi partition associated with the starting lattice along a parallel subspace to the external space (this will be published elsewhere). Accordingly, the dodecagonal quasiperiodic tiling in the present paper is obtained also by the grid method. The relevant grid is a double triangular grid which is a superposition of two triangular grids $G_{\mathrm{T}}$ and $\tilde{G}_{\mathrm{T}}$ as shown in figure 4; $G_{\mathrm{T}}$ and $\tilde{G}_{\mathrm{T}}$ are the cross sections of $\Pi_{\mathrm{HHC}} \times \mathrm{E}_2$ and $\mathrm{E}_2 \times \Pi_{\mathrm{HHC}}$, respectively, so that the axes of $\tilde{G}_{\mathrm{T}}$ are rotated by 30° from those of $G_{\mathrm{T}}$. A regular hexagon in the tiling presented in figure 2 is dual to a triple crossing point of either triangular grid; two possible orientations of the hexagons correspond to the two triangular grids. A crossing point between two lines belonging to different triangular grids is dual to a square or a thin rhombus depending on the crossing angle.

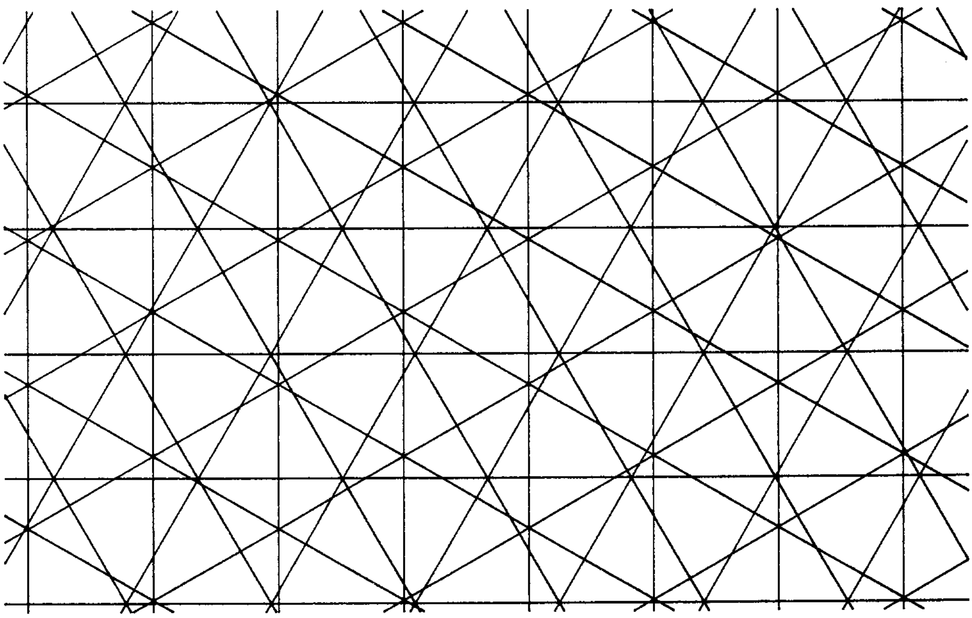

**Figure 4.** A double triangular grid, whose dual counterpart is a dodecagonal quasiperiodic tiling as given in figure 2.

Incidentally, we remark that the double honeycomb grid used by Stampfli (1986) for constructing a dodecagonal quasiperiodic tiling is nothing but a cross section of the Voronoi partition of $L_{\mathrm{HH}}$ $(= L_{\mathrm{T}} \times L_{\mathrm{T}})$ along a parallel plane to the external space; the Voronoi partition of $L_{\mathrm{T}}$ is of honeycomb structure. On the other hand, the LI class obtained by Ishihara (1987) can be derived from a double Kagomé grid which is a superposition of two Kagomé grids in 2D.

*Note added in proof.* A dodecagonal quasiperiodic tiling in Ishihara (1987) has been published subsequently (Ishihara *et al* 1988).

## References

de Bruijn N G 1981 *K. Nederl. Acad. Wetensch. Proc.* A **84** 51-56
Duneau M and Katz A 1985 *Phys. Rev. Lett.* **54** 2688-91
Gähler F 1986 *J. Physique* **C3** 115-22
Gähler F and Rhyner J 1986 *J. Phys. A: Math. Gen.* **19** 267-77
Ishihara K 1987 *Proc. Riken Symp. on the Structures of Quasi-crystals and Related Subjects* (Tokyo: Institute of Physical and Chemical Research) pp 4-7 (in Japanese)
Ishihara K, Nishitani S R and Shingu P H 1988 *Trans. ISIJ* **28** 2-6
Ishimasa T, Nissen H-U and Fukano Y 1985 *Phys. Rev. Lett.* **55** 511-3
Janssen T 1986 *Acta Crystallogr.* A **42** 261-71
Kalugin P A, Kitaev A Y and Levitov L S 1985 *JETP Lett.* **41** 145-9
Katz A and Duneau M 1986 *J. Physique* **47** 181-96
Krammer P and Neri 1984 *Acta Crystallogr.* A **40** 580-7
Niizeki K and Mitani H 1987 *J. Phys. A: Math. Gen.* **20** L405-10
Stampfli P 1986 *Helv. Phys. Acta* **59** 1260-3
Yang Q B and Wei W D 1987 *Phys. Rev. Lett.* **58** 1020-3
Zia R K P and Dallas W J 1985 *J. Phys. A: Math. Gen.* **18** L341-5